MATH 5061 Riemannian Geometry I

Q: What is "Riemannian Geometry"?

Undergrad. Diff. Geom. (~MATH 4030) $\xrightarrow{\text{continuation}}$ Riemannian Geometry (MATH 5061)

"higher dim."
"intrinsic geometry"

Curves / surfaces in $\mathbb{R}^2$ or $\mathbb{R}^3$
(curvatures)

geometry / topology / calculus / analysis, PDEs

(M^n, g)

abstract (smooth) manifold

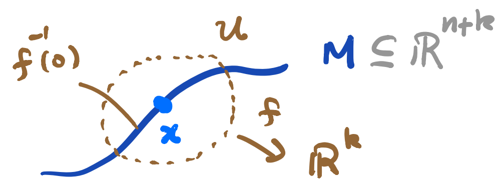
Riemannian metric

We begin with the theory of (smooth C^∞) manifolds.

(Ref: F. Warner "Foundations of Differential Manifolds and Lie Groups"
J. Lee "Introduction to Smooth Manifolds")

§ Submanifolds of $\mathbb{R}^N$

Def$^{\underline{n}}$: $M \subseteq \mathbb{R}^{n+k}$ is an n-dim'l submanifold of $\mathbb{R}^{n+k}$ (of class C^p) if $\forall\, x \in M$, $\exists$ nbd. $x \in \mathcal{U} \subseteq \mathbb{R}^{n+k}$ and a C^p-map $f : \mathcal{U} \to \mathbb{R}^k$ s.t.



(i) $df_q : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is onto $\forall\, q \in \mathcal{U}$ (i.e. f is a submersion)

(ii) $\mathcal{U} \cap M = f^{-1}(0)$

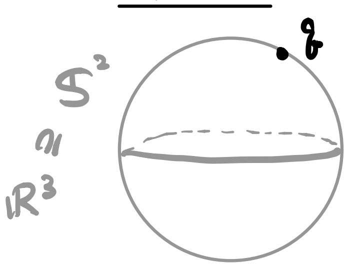
Idea: $M \overset{\text{locally}}{=}$ "regular" zero set of a C^p (vector-valued) function

Here, $k = \operatorname{codim} M$, $n = \dim M$.

When $k = 1$, we say M is a hypersurface.

Examples of submfd in $\mathbb{R}^N$

(a) Sphere $\quad \mathbb{S}^n := \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 = 1 \}$



$$f(x,y,z) = x^2 + y^2 + z^2 - 1$$

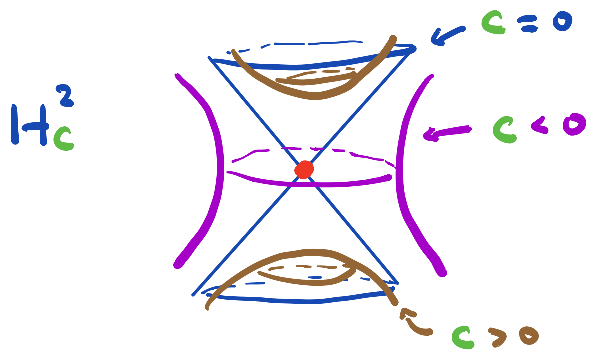
$$\Rightarrow \quad f : \mathbb{R}^3 \to \mathbb{R} \quad C^\infty \quad \text{and} \quad f^{-1}(0) = \mathbb{S}^n$$

$$df_q = (2x, 2y, 2z) \neq 0 \quad \forall q \in \mathbb{S}^n$$

(b) Hyperboloid $\quad H_c^n := \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 - x_1^2 - \dots - x_n^2 = c \}$

is a C^∞-submfd in $\mathbb{R}^{n+1}$ when $0 \neq c \in \mathbb{R}$.

$n = 2$:



(c) Torus $\quad T^n := \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \dots + |z_n|^2 = 1 \}$

is a C^∞-submanifold of $\mathbb{C}^n \cong \mathbb{R}^{2n}$

($\dim = n$, $\operatorname{codim} = n$)

(d) $\quad SO(n) := \{ A \in M_n(\mathbb{R}) \mid A^t A = I,\ \det A = 1 \}$

is a C^∞-submfd of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ $\quad \left(\dim SO(n) = \frac{n(n-1)}{2} \right)$

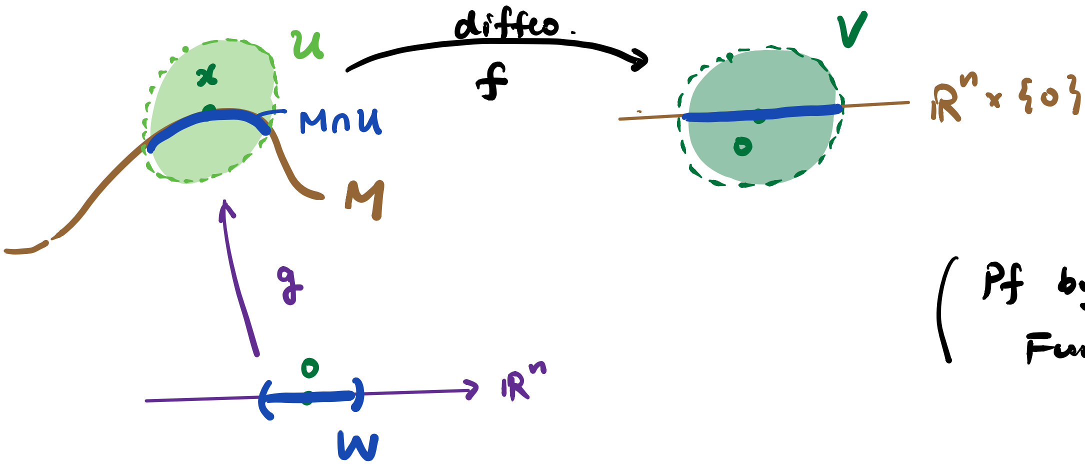
Prop: TFAE:

(i) $M^n \subseteq \mathbb{R}^{n+k}$ is a C^p-submanifold

(ii) $\forall\, x \in M$, $\exists$ nbd ${}^{x\in}\mathcal{U} \subseteq \mathbb{R}^{n+k}$, ${}^{0\in}V \subseteq \mathbb{R}^{n+k}$

and a C^p-diffeomorphism $f : \mathcal{U} \to V$

s.t. $f(\mathcal{U} \cap M) = V \cap (\mathbb{R}^n \times \{0\})$



(Pf by Implicit Function Theorem)

(iii) $\forall\, x \in M$. $\exists$ nbd $x \in \mathcal{U} \subseteq \mathbb{R}^{n+k}$ and $0 \in W \subseteq \mathbb{R}^n$

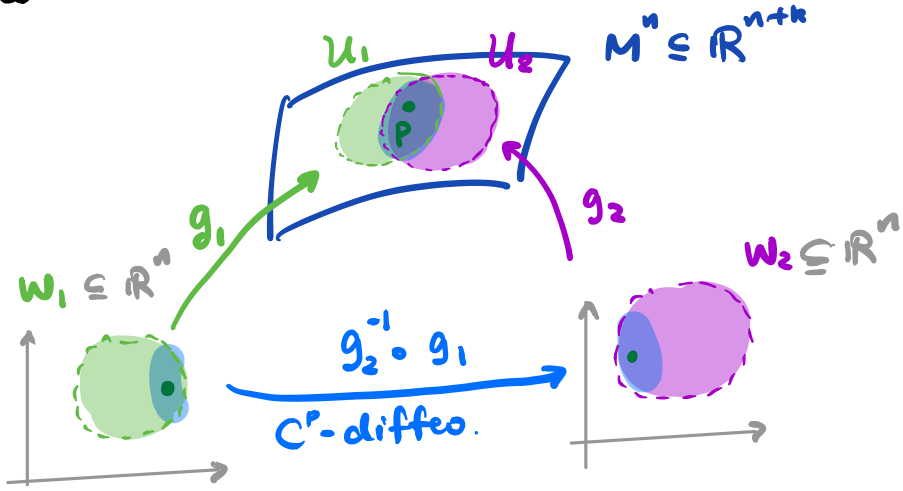
and a C^p-map $g : W \to \mathbb{R}^{n+k}$

s.t. g is a homeomorphism onto its image $g(W) = M \cap \mathcal{U}$

with dg_0 is 1-1. ("local parametrization / chart")

Remark: For any two such charts $g_i : W_i \to \mathbb{R}^{n+k}$, $i = 1, 2$, as in (iii)

then the transition maps $g_2^{-1} \circ g_1$ is a C^p-diffeomorphism



§ Abstract Manifolds

Idea: "n-manifolds" $\overset{\text{"locally"}}{\approx}$ open subsets of $\mathbb{R}^n$

described by "compatible" charts into $\mathbb{R}^n$

ASSUME: M Hausdorff, "paracompact" topological space

[$\Rightarrow \exists$ "partition of unity"]

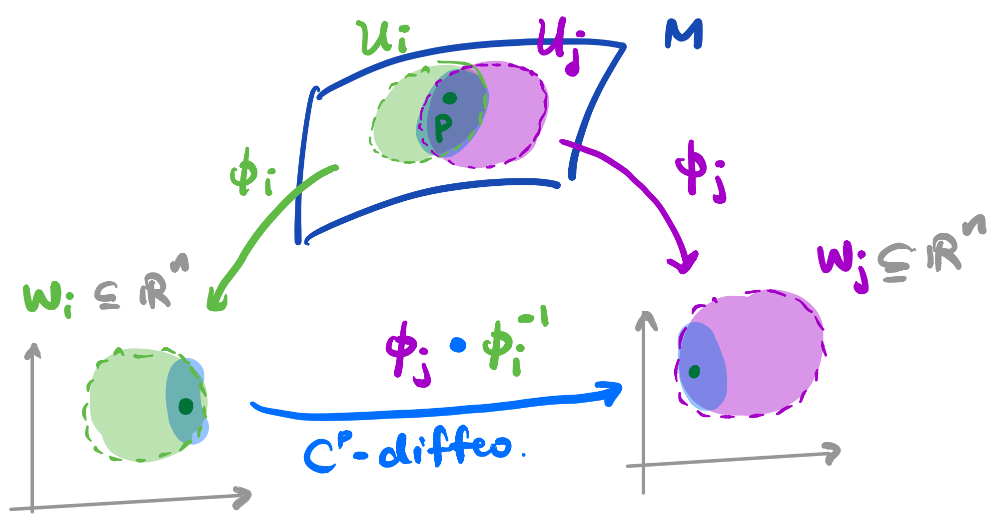
Defⁿ: A C^p-atlas on M is a collection of charts $\{(U_i, \phi_i)\}_{i\in I}$

s.t. (i) $\{U_i\}_{i\in I}$ forms an open cover of M

(ii) $\phi_i : U_i \to W_i \overset{\text{open}}{\subseteq} \mathbb{R}^n$ are homeomorphisms $\forall i \in I$

and the transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \text{ are } C^p\text{-diffeo.}$$



Defⁿ: $\{(U_i, \phi_i)\}_{i\in I} \sim \{(V_j, \psi_j)\}_{j\in J}$ if their union is an atlas

Def^{n}: An equivalence class of C^p-atlas on M is called a **differentiable structure** (of class C^p) on M.

A **differential manifold** consists of a Hausdorff, paracompact topological space M together with an atlas $\{(U_i, \phi_i)\}_{i \in I}$.

Remark: M connected $\Rightarrow$ $\dim M = n$ well-defined

(by "invariance of domain")

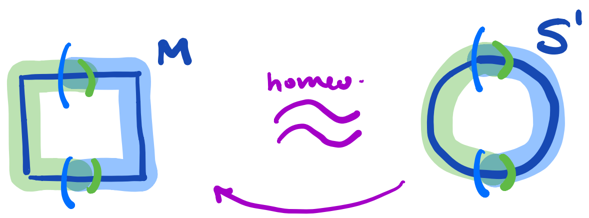
ASSUME: M^n connected, smooth (i.e. C^∞) manifold

Def^{n}: $N \subseteq M^n$ is a **submanifold** if $\forall p \in N$, $\exists$ chart $(\mathcal{U}, \phi)$ of M s.t. $p \in \mathcal{U}$ and $\phi(N \cap \mathcal{U}) \subseteq \mathbb{R}^n$ is a submfd.

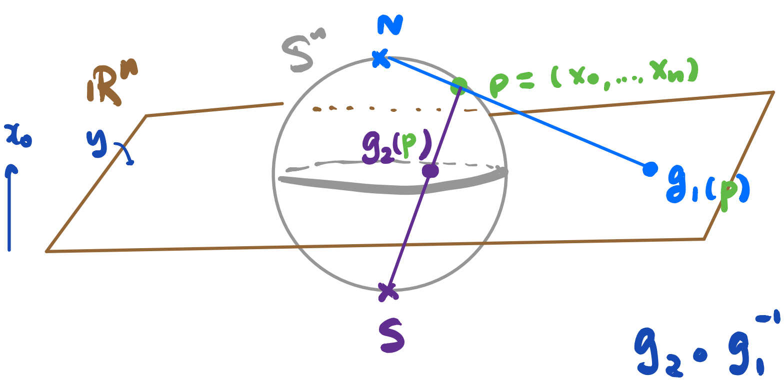
Examples of abstract manifolds

(a) submfds of $\mathbb{R}^{n+k}$

(b) $\exists$ C^∞-structure on a square (not inherited from $\mathbb{R}^2$)



(c) $S^n \subseteq \mathbb{R}^{n+1}$ can be covered by only two charts:

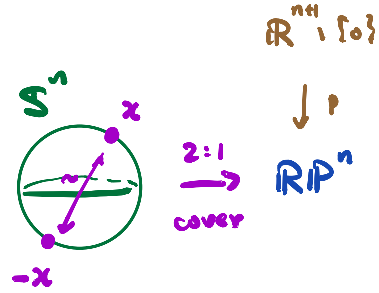


$$g_1(x_0, \dots, x_n) = \frac{(x_1, \dots, x_n)}{1 - x_0} \qquad p \neq N$$

$$g_2(x_0, \dots, x_n) = \frac{(x_1, \dots, x_n)}{1 + x_0} \qquad p \neq S$$

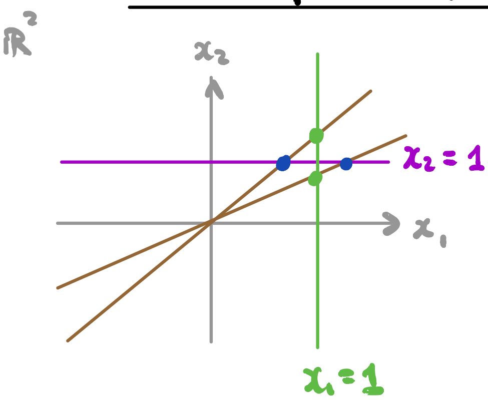
$$g_2 \circ g_1^{-1}(y) = \frac{y}{\|y\|^2}$$ diffeo. on $\mathbb{R}^n \setminus \{0\}$.

(d) Real Projective Space $\mathbb{R}P^n$.



$$\mathbb{R}P^n := \mathbb{R}^{n+1}\setminus\{0\} \Big/ \underset{(\lambda\neq 0)}{x \sim \lambda x} = S^n / \{\pm 1\}$$

A description of C^∞-structure on $\mathbb{R}P^n$:



$\Phi_1 : \{x_1 \neq 0\} \longrightarrow \mathbb{R} \qquad \Phi_1(x_1, x_2) = \dfrac{x_2}{x_1}$

$\Phi_2 : \{x_2 \neq 0\} \longrightarrow \mathbb{R} \qquad \Phi_2(x_1, x_2) = \dfrac{x_1}{x_2}$

Note: $\Phi_i(\lambda x_1, \lambda x_2) = \Phi_i(x_1, x_2) \quad (\lambda \neq 0)$

$\rightsquigarrow$ Φ_i are well-defined on $\mathbb{R}P^1$.

and they form a chart on $\mathbb{R}P^1$

(Ex: check this!)

(e) Replace $\mathbb{R}$ by $\mathbb{C}$ $\rightsquigarrow$ Complex Projective Space $\mathbb{C}P^n$

(dim = 2n)

Def$^{\underline{n}}$: M is orientable if $\exists$ atlas $\{(U_i, \phi_i)\}_{i\in I}$ s.t.

all transition maps are orientation-preserving

[i.e. $\det(d(\phi_j \circ \phi_i^{-1})) > 0$].

Examples: S^n is orientable BuT $\mathbb{R}P^n$ is NOT when n is even.

§ Smooth Maps between manifolds

Let M^m, N^n be smooth manifolds.

<u>Def$^{\underline{n}}$</u>: A cts map $f: M \to N$ is smooth

if $\forall x \in M$, $\exists$ charts (U, ϕ) for $x \in M$

and chart (V, ψ) for $f(x) \in N$

s.t. $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

i.e.

$$\begin{tikzcd} M \supseteq \overset{x}{\underset{\cap}{}} U \arrow[r, "f"] \arrow[d, "\phi"'] & V \ni f(x) \subseteq N \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) \arrow[r, "\psi \circ f \circ \phi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

<u>Example</u>: $M^n \subseteq \mathbb{R}^{n+k}$ submfd <u>and</u> $F: \mathbb{R}^{n+k} \to \mathbb{R}$ smooth

$\Rightarrow$ $F|_M : M \to \mathbb{R}$ is a smooth map between manifolds

<u>Def$^{\underline{n}}$</u>: A smooth map $f: M \to N$ is called an immersion at $p \in M$

if $\exists$ charts $(\overset{p}{\cap}U, \phi)$ for M, $(\overset{f(p)}{\cap}V, \psi)$ for N s.t.

$d(\psi \circ f \circ \phi^{-1})$ is 1-1 at $\phi(x)$.

<u>Remark</u>: submersion / local diffeo. if it is onto / bijective

<u>Def$^{\underline{n}}$</u>: $f: M \to N$ diffeomorphism if f is bijective and

both f, f^{-1} are smooth.

<u>Exercise</u>: $\mathbb{CP}^1 \overset{\text{diffeo.}}{\cong} S^2$.

Locally (by IFT), in some local coord.,

immersion: $(m \leq n)$ $\qquad f(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$

submersion: $(m \geq n)$ $\qquad f(x_1, \dots, x_m) = (x_1, \dots, x_n)$

Def$^{\underline{n}}$: $f: M \to N$ embedding
iff f is an immersion; and $f: M \to f(M)$ homeomorphism

Example: $M = (0,1)$, $N = \mathbb{R}^2$; $f: (0,1) \longrightarrow \mathbb{R}^2$